VOL. 115 NO. 2 MAY 1989

ISSN 0733-9453
CODEN: JSUED2

Journal of Surveying Engineering

GPS Satellite Surveys and Vertical Control[a]

By David B. Zilkoski[1] and Larry D. Hothem,[2] Member, ASCE

Abstract: Analysis of Global Positioning System (GPS) survey data has shown that GPS can be used to establish precise relative positioning in a three-dimensional system. The results of many tests and operational projects have clearly shown that GPS survey methods can replace classical horizontal terrestrial survey methods. Comparable accuracies have also been achieved for GPS-derived ellipsoid height differences. The problem of converting these ellipsoid height differences to orthometric height differences remains to be resolved. Can the accuracies achieved for these orthometric height differences provide a viable alternative to classical geodetic leveling techniques? Some results of analyses performed by the authors in estimating orthometric heights from GPS surveys indicate that with appropriate planning, consideration of GPS survey specifications for connection to bench marks, proper field observing procedures, and proper strategy for estimating geoid heights and final orthometric height values, it is possible to use GPS survey methods to estimate orthometric heights to meet a wide range of engineering requirements for vertical control. Therefore, another question needs to be addressed. What are the accuracy requirements of most engineering and mapping applications? This is best answered by users of the data and will influence how much effort should be directed toward determining more accurate geoid heights.

Introduction

Since early 1983, the National Geodetic Survey (NGS) has performed control survey projects in the United States using satellites of the Global Positioning System (GPS). These GPS surveys have met the requirements of many users, e.g., analysis of subsidence (Strange 1985) and connection of control points to the National Geodetic Reference System (Hothem et al. 1984).

Analysis of GPS survey data has shown that GPS can be used to establish precise relative positions in a three-dimensional Earth-centered coordinate system. GPS carrier phase measurements are used to determine vector base lines in space where the components of the base line are expressed in terms of Cartesian coordinate differences ($\Delta X, \Delta Y, \Delta Z$) (Remondi 1984). These vector base lines can be converted to distance, azimuth, and ellipsoidal height differences (dh) relative to a defined reference ellipsoid. Orthometric height differences (dH) can then be obtained from ellipsoid height differences by subtracting the geoid height differences (dN):

$$dH = dh - dN \quad (1)$$

(Note: this equation is approximate; the error is always very small and is considered to be insignificant.)

[a]Presented at the GPS-88 Engineering Applications of GPS Satellite Technology at Nashville, Tennessee, May 11–14, 1988.

[1]Vertical Network Branch, National Geodetic Survey, Charting and Geodetic Services, NOS, NOAA, Rockville, MD 20852.

[2]Gravity, Astronomy, and Space Geodesy Branch, National Geodetic Survey, Charting and Geodetic Services, NOS, NOAA, Rockville, MD.

Note. Discussion open until October 1, 1989. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on May 24, 1988. This paper is part of the *Journal of Surveying Engineering*, Vol. 115, No. 2, May, 1989. ©ASCE, ISSN 0733-9453/89/0002-0262/$1.00 + $.15 per page. Paper No. 23496.

The results of many tests and operational projects have clearly shown that GPS survey methods can replace classical horizontal terrestrial survey methods. Comparable accuracies have also been achieved for GPS-derived ellipsoid height differences. However, there remains the problem of converting these ellipsoid height differences to orthometric height differences. Can the accuracies achieved for these orthometric height differences provide a viable alternative to classical geodetic leveling techniques?

There are several reports documenting the technical aspects of GPS relative positioning (Remondi 1984; King et al. 1985; Wells 1986; NOAA 1985; and DMA 1986) and geoid height determinations (Schwarz et al. 1987; Kearsley 1984; Kearsley 1985; Kearsley 1986a; Kearsley 1986b; Fury 1986; Rapp 1981; Engelis et al. 1984; Moritz 1983; Vanicek and John 1983; Tscherning 1974; and Heiskanen and Moritz 1981). It is clear that GPS-derived orthometric heights will have a major impact on the surveying community in the future. There are, however, several factors that must be investigated and documented before GPS-derived orthometric heights can be used routinely by the surveying community. This report gives some results of the analyses performed by the authors in comparing orthometric heights determined by differential leveling techniques to orthometric height differences from GPS surveys and predicted geoid height differences. Factors that need to be considered for GPS-derived orthometric heights are discussed.

Heights and Height Differences

Orthometric heights (H) are referenced to an equipotential surface, e.g., the geoid. The orthometric height of a point on the Earth's surface is the distance from the reference surface to the point, measured along the plumb line normal to the geoid. Ellipsoid heights (h) are referenced to a reference ellipsoid. The ellipsoid height of a point is the distance from the reference ellipsoid to the point, measured along the line that is normal to the ellipsoid. The difference between an ellipsoid height and an orthometric height is defined as the geoid height (N) (to a sufficient approximation). Fig. 1 depicts these three different height systems.

Please note that, for purposes of simplicity, the H and h in Fig. 1 are shown to be along a common vertical. Actually, H is normal to the geoid and h is normal to the ellipsoid (the difference being the deflection of the vertical). This would cause a very small error that is considered insignificant compared to present uncertainties of geoid height difference estimates.

Several error sources that affect the accuracy of orthometric, ellipsoid, and geoid height values are generally common to nearby points. Because these error sources are common, the uncertainty of height differences between nearby points is significantly smaller than the "absolute" heights at a point.

Ellipsoidal height differences (dh) can be determined from GPS phase measurements with 1-sigma uncertainties that are typically +/− (0.5 cm + 1–2 ppm) (DMA 1986 and NOAA 1985). With improved orbit determination techniques, dual frequency carrier phase data, and improved antenna designs, uncertainties approaching +/− (0.2 cm + 0.01–0.1 ppm) may be achieved for dh values. Table 1 summarizes repeat GPS ellipsoid height difference measurements determined from data obtained during the testing of GPS survey equipment by the Federal Geodetic Control Committee (FGCC) on the test network located in the vicinity of Washington, D.C. Geoid height

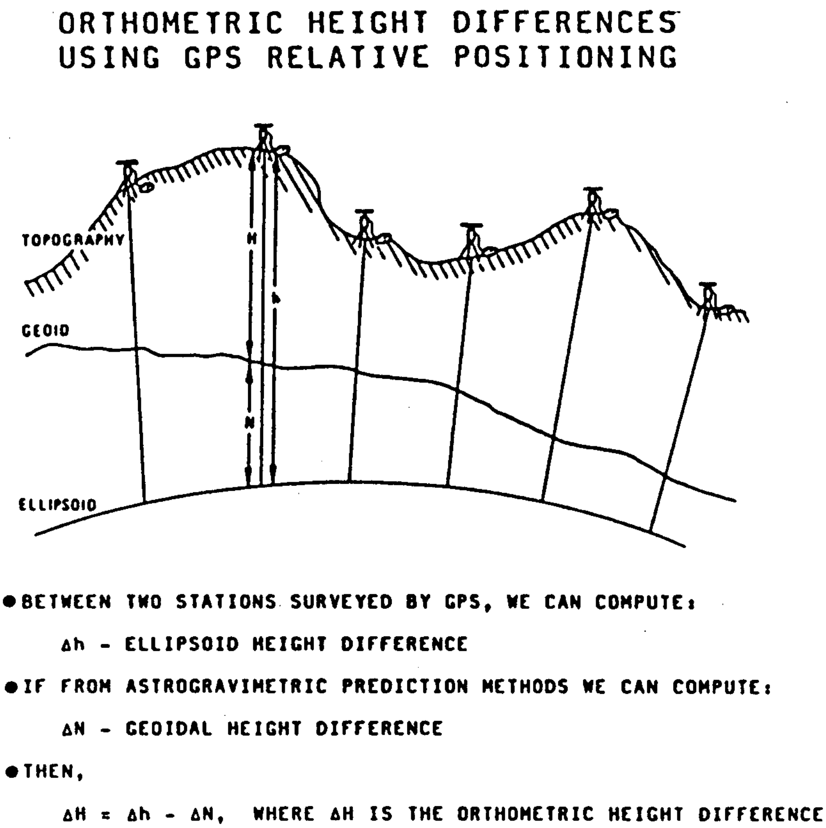

FIG. 1. Orthometric Height Differences Using GPS Relative Positioning

differences (dN) in the United States can be determined from gravity data and Stokes' integral method, or from astrogravimetric data and least squares collocation methods with uncertainties that are typically 1–10 cm for distances of as much as 20 km and 5–20 cm for distances from 20 to 50 km (Fury 1986 and R. Fury, NGS, personal communication 1988). The lower value for the uncertainties has been demonstrated in tests in several regions of the United States. Larger uncertainties can be expected in other areas,

TABLE 1. Summary of Repeat GPS Ellipsoid Height Difference Measurements, FGCC Test Network, Washington, D.C.

Stations		Distance	Observing span range	Number of repeat	Standard Deviation of Mean	
From (1)	To (2)	(km) (3)	(min) (4)	measurements (5)	cm (6)	Prop. part (7)
NBS1	NBSO	0.18	18 to 162	12	0.4	1:45,000
NBS1	NBS3	0.49	18 to 122	15	0.6	1:82,000
NBS1	NBS4	0.75	18 to 164	11	1.0	1:75,000
NBS1	ORM1	1.32	18 to 149	20	1.2	1:110,000
NBS1	ATHY	7.39	13 to 164	17	1.2	1:620,000
NBS1	OPTK	17.55	25 to 165	12	2.5	1:700,000
NBS1	GORF	35.66	25 to 165	10	3.8	1:940,000
NBS1	ASTW	104.35	21 to 170	4	17.3	1:600,000

Note: Repeat measurements derived from GPS observations collected with six different systems during test surveys conducted between January 1983 and November 1988.

depending on the density of the gravity network and uncertainties in the determination of gravity anomalies.

When high-accuracy field procedures are used, orthometric height differences can be computed from measurements of precise geodetic leveling with an uncertainty of less than 1 cm over a 50–kilometer distance. Lower accuracies are achieved when third–order leveling methods are employed. Depending on the accuracy requirements, GPS surveys and present geoid prediction models can be employed as an alternative to classical geodetic leveling methods, but are not usually as accurate. The obvious limiting factor is the accuracy of estimating geoid height differences. The models used to estimate geoid heights are usually too generalized to accurately represent the local relief of the geoid. There are, however, a wide range of existing engineering projects for which GPS surveys may meet vertical control requirements.

Error Sources

As in other geodetic techniques, the accuracy of the results depends on the significance of error sources. Table 2 lists some error sources that may affect the accuracy of GPS-derived orthometric height differences.

The most significant error source is the uncertainty in computing geoid height differences that varies significantly throughout the United States. The magnitude of the uncertainties depends mostly on the variability of the gravity field, the density and distribution of gravity observations, and the accuracy in gravity data reductions. From this study, uncertainties of 5–15 cm were typical, except for GPS projects performed in mountainous regions where uncertainties of more than 1 m were detected. As stated above, the models used to estimate geoid heights are usually too generalized to accurately represent the local relief of the geoid.

The error sources associated with GPS data can be minimized by adhering to appropriate specifications and procedures (Hothem 1988). Some examples include using two–frequency receivers and accurate surface weather measurements to reduce the effects of atmospheric refraction, selecting appropriate sites to optimize connections to points with known precise orthometric height differences, minimizing the effects of obstructions, accurately measuring the height of the antenna phase center before and after data collection, and selecting observing periods with optimal satellite geometry and number. By adhering to the appropriate specifications and procedures, the effects of error sources influencing GPS can be minimized in order to obtain ellipsoid height differences with uncertainties typically better than +/− (0.5 cm + 1–2 ppm). Thus, the limiting factor in estimating GPS-derived orthometric heights to meet the accuracy requirements for most engineering and land surveys is the accuracy of estimating geoid height differences.

Methods Used by NGS to Estimate Geoid Heights

NGS has used three techniques for the calculation of geoid heights: (1) Stokes' classical method (Fury 1984; and Fury 1986); (2) least squares collocation (Tscherning 1974); and (3) computation using the Earth's gravity field represented by spherical harmonics.

The classical method of Stokes uses only gravity data in estimating the undulation of the geoid about a reference ellipsoid:

TABLE 2. Error Sources That May Affect the Accuracy of GPS-Derived Orthometric Height Differences

Factor (1)	Error source (2)
Geoid height difference	Prediction method
	Distribution and accuracy of data used in predictions
GPS antenna setup	Height measurements
	Correct identification of bench mark
Vertical network control	Accuracy of orthometric heights
	Internal consistency
	Vertical deformation
	Misidentified bench mark
GPS data	Antenna
	Phase center instability
	Multipath effects
	Different antenna designs
	Atmospheric refraction
	Ionospheric
	Tropospheric
	Orbital coordinate data
	Broadcast (predicted)
	Precise
	Period of the observing session
	Processing software
	Method—single, double, or triple difference processing
	Mode—single, multiple, or network base line solutions
	Orbit data—fixed versus adjusted refinement
	Satellite constellation
	Status—health
	Number available
	Geometric distribution
	Spacing between stations
	Survey site obstructions

$$N = \frac{R}{4\pi\bar{g}} \int_0^{2\pi} \int_0^{\pi} \Delta g(\alpha, \psi) S(\psi) \sin\psi \, d\psi \, d\alpha \quad (2)$$

In this expression ($\bar{g}$) = an average (global) value of gravity; R = the mean radius of the earth; (Δg) represents free air gravity anomalies which are derived from measurements; $S(\psi)$ is the familiar Stokes function; and N is the calculated geoid height at a given station.

Stokes' formula is evaluated through the application of fast numerical algorithms that inevitably involve various degrees of approximation. Other critical aspects of this technique are the method by which gravity anomalies (Δg) are computationally derived and, of course, the assumption that the density and distribution of observations are sufficient to adequately model the geoid's slope and its changes in slope.

Since the numerical algorithms were developed to meet the objectives of the North American (horizontal) Datum of 1983 readjustment project, they do not, in general, yet fulfill centimeter-level prediction requirements (Fury

1984). In addition, the accuracy of estimated geoid heights varies as a function of quality and quantity of observed data, especially in areas with steep geoid slope.

The second technique used by NGS to estimate geoid heights utilizes least squares collocation (Tscherning 1974; and Fury 1986):

$$N = \bar{C}^{T}_{N,\Delta g}[C_{\Delta g,\Delta g}]^{-1}\overline{\Delta g}$$

In this expression, ($\overline{\Delta g}$) represents a vector of gravity anomalies; ($C_{\Delta g,\Delta g}$) is a covariance matrix of the gravity anomalies; and ($C_{N,\Delta g}$) is a cross-covariance vector between geoid heights and gravity anomalies.

There are three basic factors which affect the accuracy of computed geoid heights using least squares collocation: (1) The extent to which the covariances reflect the data; (2) the strategies of data selection; and (3) the density and distribution of gravity anomalies. The distinctly different characteristic of this technique, as compared to that of Stokes', is the ability to utilize a combination of measurements in addition to gravity: deflections of the vertical and geoid heights derived from geometric measurements and leveling; e.g., estimated geoid heights using GPS data and leveling data. This formulation also provides estimates of the accuracy of computed geoid heights, in contrast to the Stokes' technique.

Geoid heights can be conveniently computed from coefficients of a spherical harmonic series (Heiskanen and Moritz 1981) that may be derived from globally distributed gravity anomalies, data from satellite altimetry, satellite-to-satellite or satellite-to-Earth tracking data, and/or the combination of any of the above. The global gravity field, i.e., Earth model, defined in terms of spherical harmonics, must be expanded to high degree and order (e.g., n = 180, n = 360 or higher) in order to achieve a useful resolution in geoid detail. Presently available geoid models are generally inadequate to provide the details of local geoid surfaces required for accurate estimates of orthometric height determinations using GPS. In other words, the models are too generalized to accurately represent the local relief of the geoid. NGS has used the n = 180 field of Rapp (1981) and the n = 360 field of Rapp and Cruz (1986).

The full potential of these techniques has not been fully exploited. The demonstrated requirements for higher accuracies by the surveying community will be a strong motivation to improve the numerical techniques presently employed, as well as to institute a gravity compilation/observation program for a more complete gravity data base.

Data Evaluation

An important aspect of any geodetic positioning technique is to ensure that all data outliers have been removed from the data. The design of the network can be very helpful when analyzing the data.

GPS results can be evaluated by analyzing network loop misclosures, repeat base line differences, and least squares adjustment results. The design of the network should be such that there are enough redundant observations to detect data outliers.

It was mentioned previously that the largest contribution to the error budget is uncertainty in geoid height difference estimates. Therefore, it is important to evaluate these estimates. Since the slope of the geoid can change significantly between "widely" spaced monuments, it is necessary to perform a detailed study of the density and distribution of observed gravity values (or free-air anomalies) to determine the slope and changes in slope. The distribution of known orthometric heights is extremely important in verifying geoid height differences. Vertical control stations (bench marks) should be strategically located throughout the network in order to determine the geoid's slope and its changes in slope (flatness).

There are four major items that must be considered when orthometric heights of bench marks obtained from leveling data are used to evaluate GPS-derived orthometric height results. First, and most important, is that all orthometric heights must be referenced to the same datum; e.g., the National Geodetic Vertical Datum of 1929 (NGVD 29). Second, the network must be designed in such a manner that bench marks which have been disturbed or influenced by vertical crustal motion will be detected during the analyses. Third, all leveling data used to establish the heights should be corrected for known systematic effects. Last, the latest and/or most accurate data available should be used to estimate the orthometric height differences between monuments.

Results

Precise geodetic leveling data were used to estimate orthometric height differences between monuments. These height differences provided the standards used for the comparisons. GPS-derived orthometric height differences estimated from ellipsoid height differences and geoid height differences were then subtracted from the differential leveling results to obtain the comparison (denoted as *ddH* in the tables). Tables 3 through 11 list some results of analyses performed by the authors in estimating GPS-derived orthometric heights.

TABLE 3. Comparison of Orthometric Height Differences, Summit County, GPS Survey Project, 1983

Stations		Distance	Approximate elevation	$(ddH)/(\Delta H_{\text{Leveling}}$ Minus $\Delta H_{\text{GPS}})$[a]	
From (1)	To (2)	(km) (3)	difference (m) (4)	cm (5)	proportional (6)
Otis	Airport	2.2	−30	0.1	1:2,200,000
Airport	Square	1.3	37	0.3	1:430,000
W. Pond	H 179	10.6	−4	−4.9	1:220,000
U 177	Horst Azi	11.4	72	−2.6	1:440,000
U 177	B 10	33.1	42	3.6	1:920,000
N 177	H 179	11.1	4	8.7	1:130,000
39 RH	B 10	19.9	−27	−2.5	1:800,000
Square	S 337	3.2	−23	−2.5	1:128,000
N 177	U 177	7.2	1	−1.5	1:480,000
39 RH	S 337	7.2	−30	4.2	1:170,000

Mean = 0.3 cm
Standard deviation = +4.1 cm

[a]Geoid height differences predicted with MCANAL—least squares collocation.
Note: Range for station elevations: 288 to 355 meters; range for geoid height differences: −34 to 14 cm.

TABLE 4. Comparison of Orthometric Height Differences, Memphis, Tennessee-Rena Lara, Mississippi, GPS Survey Project, 1984

Stations		Distance	Approximate elevation	$(ddH)/(\Delta H_{Leveling}$ Minus $\Delta H_{GPS})$[a]	
From (1)	To (2)	(km) (3)	difference (m) (4)	cm (5)	proportional (6)
RENA 2	LAKE	8.42	−0.2	−0.1	0.1 ppm
RENA 2	J 346	4.14	−0.6	1.2	1:350,000
RENA 2	B 346	6.37	1.6	1.0	1:640,000
B 346	Z 345	2.93	0.3	−1.1	1:270,000
Z 345	MCWILLIAMS[a]	4.22	−1.3	−1.6	1:260,000
T 345	MCWILLIAMS[a]	4.61	−1.5	0.0	—
T 345	N 345	5.94	1.0	1.5	1:400,000
N 345	ADEHO 2[a]	4.38	−2.9	0.9	1:490,000
ADEHO 2[a]	G 345	5.76	5.4	1.1	1:520,000
G 345	B 345	5.57	−0.4	1.2	1:460,000
B 345	J 344	6.72	1.0	−0.5	0.7 ppm
J 344	W 349	5.54	0.1	6.6	1:84,000
W 349	AUSTIN 2	5.20	−5.4	2.5	1:210,000
AUSTIN 2	P 349	4.04	6.3	−0.2	0.5 ppm
P 349	M 349	4.00	−0.2	−0.1	0.3 ppm
M 349	C 349	9.96	1.8	0.8	0.8 ppm
C 349	ABB	4.21	1.2	−2.3	1:180,000
ABB	V 348	5.52	0.7	4.3	1:130,000
V 348	R 348	7.27	0.2	4.7	1:150,000
R 348	L 348	5.89	0.4	3.6	1:160,000
L 348	D 348	7.72	1.8	3.3	1:230,000
D 348	F 348	6.75	−1.0	−4.9	1:140,000
D 348	A 348	4.13	−5.0	2.3	1:180,000

Mean = 1.0 cm
Standard deviation = +2.5 cm

[a]Geoid height differences predicted with MCANAL—least squares collocation.
[b]Antenna mounted on 40-foot tower.
Note: Range for station elevations: 57 to 73 m; range for geoid height differences: −10 to 9 cm.

Referring to Tables 3 through 5 it is clear that in many regions of the United States, orthometric heights can be estimated using GPS and gravity data with uncertainties of 5–15 cm. However, looking at results from GPS projects performed in mountainous regions (see Tables 6 and 7), it is clear that in some regions of the United States, GPS-derived orthometric heights may be in error by more than 1 m. This is unacceptable and appropriate procedures must be followed in order to detect and reduce these large uncertainties. This will be addressed in more detail in the next section.

Table 3 lists the results from a GPS project performed in Summit County, Ohio, an area with relatively flat terrain. Fig. 2 depicts the network design of the project. Some of the GPS-derived orthometric height estimates meet FGCC section closure tolerances (FGCC 1984) for first-order, class I leveling [3.0 mm × SQRT(km)]; e.g., Otis to Airport, *ddH* = 1 mm in 2.2 km (allowable = 4.4 mm); and Airport to Square, *ddH* = 3 mm in 1.3 km (allowable = 3.4 mm).

This is encouraging, but it must also be pointed out that in the same project, there are estimates which do not even meet FGCC section closures for

TABLE 5. Comparison of Orthometric Height Differences, FGCC Test Network Surveys, Washington, D.C.

Stations		Distance (km)	Approximate elevation difference (m)	$(ddH)/(\Delta H_{Leveling}$ Minus $\Delta H_{GPS})^{a,b}$	
From (1)	To (2)	(3)	(4)	cm (5)	proportional (6)
NBS1	NBSO	0.18	2	−0.7	1:26,000
NBS1	NBS3	0.49	−1	−0.9	1:54,000
NBS1	NBS4	0.75	−11	−1.0	1:75,000
NBS1	ORM1	1.32	15	−1.0	1:130,000
NBS1	ATHY	7.39	−2	2.3	1:320,000
NBS1	OPTK	17.55	−26	2.4	1:730,000
NBS1	GORF	35.66	−86	−2.5	1:1,430,000
NBS1	ASTW	104.35	−70	−0.5	1:2,090,000

[a]Geoid height differences predicted with MCANAL—least squares collocation.
[b]Based on mean of differences for base lines included in Table 1.
Note: Range for station elevations: 52 to 154 m; Range for geoid height differences: 0.1 to 61 cm.

third-order leveling [12.0 mm × SQRT(km)]; e.g., White Pond to H 179, *ddH* = 49 mm in 10.6 km (allowable = 39.1 mm); N 177 to H 179, *ddH* = 87 mm in 11.1 km (allowable = 40.0 km); and 39 RH to 337, *ddH* = 42 mm in 7.2 km (allowable = 32.2 mm). These estimates may still meet the requirements of some engineering or mapping projects, but surveyors performing third-order leveling could not use these values to verify their results.

It should be noted that the dh_{GPS} values for the Summit County project were obtained from triple difference solutions. If the GPS data were reprocessed with single or double difference solutions, the results would probably improve slightly. This shows the importance of documenting which processing method was used to reduce the GPS data. One method may be good enough for horizontal control computations, but not accurate enough for estimating vertical control values.

Another consideration is the inconsistency of some comparisons. Looking at Table 3, line segments U 177 to Horst Azi and N 177 to H 179 are of approximately the same length, but their *ddH* values are considerably different. Similarly, N 177 to U 177 and 39 RH to S 337 are the same length, but their *ddH* are significantly different. In addition, *ddH* for line segment N 177 to U 177 falls within FGCC section closures tolerances for second-order, class I [6.0 mm × SQRT(km)], while *ddH* for the segment 39 RH to S 337 does not even meet FGCC section closure tolerances for third-order leveling [12.0 mm × SQRT(km)]. Due to these inconsistencies, users must be concerned about the fluctuations in accuracy of GPS-derived orthometric heights.

The distribution of differential leveling orthometric heights is critical in estimating the accuracy of GPS-derived orthometric heights. Referring to Fig. 2, it is obvious that bench marks with leveling orthometric heights located in the center of the network would help in estimating the accuracies

TABLE 6. Comparison of Heights in Boulder County, Colorado, GPS Survey Project

Station name (1)	$H_{Leveling}$ (m) (2)	$H_{Leveling}$ MINUS H_{GPS}: Adjustment 1[a] Set 1[c] (cm) (3)	Adjustment 1[a] Set 2[d] (cm) (4)	Adjustment 2[b] Set 1[c] (cm) (5)	Adjustment 2[b] Set 2[d] (cm) (6)	Geoid Height Set 1 (m) (7)	Geoid Height Set 2 (m) (8)
Buff	1,637.810	15.9	−5.1	13.9	3.3	−19.5	−15.5
Bunce	2,504.696	35.5	−152.5	−7.4	−8.1	−16.4	−14.1
Copeland	2,539.501	58.8	−181.2	Constrained	Constrained	−15.5	−13.7
Evergreen[e]	2,211.90	−194.7	−281.7	−218.2	−254.2	−18.1	−14.8
F 391	1,439.091	9.8	−56.2	23.7	−8.4	−21.7	−18.2
F 414	1,404.617	74.9[f]	−63.1	87.8[f]	−8.4	−21.4	−18.6
Ft. Collins 2	1,526.307	−51.5	−64.5	−27.9	−3.0	−20.1	−16.0
Granby[e]	2,491.30	−74.3	−259.3	−196.0	100.8	−14.8	−12.5
Line	1,621.344	−15.0	12.0	−14.0	−0.8	−21.2	−16.7
J 320 Reset	1,532.433	−15.2	−8.2	−19.8	15.0	−20.4	−16.1
NOAA	1,697.176	7.7	−8.3	Constrained	Constrained	−19.5	−15.5
Peak	2,784.980	44.7	−162.3	Constrained	Constrained	−15.9	−13.8
Platteville	1,497.357	−0.2	−43.2	Constrained	Constrained	−21.8	−18.0
T 320 Reset	1,704.890	−12.6	−31.6	−16.8	8.9	−19.4	−15.4
TT 23 USGS	1,797.943	Constrained	Constrained	Constrained	Constrained	−19.8	−15.6
U 356	1,609.930	−43.0	−72.0	Constrained	Constrained	−19.6	−15.7
Ware GPS	1,603.729	17.6	0.6	18.2	10.3	−19.5	−15.5
X 402	1,489.871	−51.3	−52.3	−21.0	9.5	−20.6	−16.4

[a]Adjustment 1—Adjusted heights (H_{GPS}) obtained from 3-D minimum constraint least squares adjustment.

[b]Adjustment 2—Adjusted heights (H_{GPS}) obtained from 3-D least squares adjustment solving for geoidal tilt and scale.

[c]Set 1–Geoid heights estimated using gravity data and Stokes' integration method as documented by Fury (1984).

[d]Set 2—Geoid heights based on Rapp's 360 × 360 model (OSU86F) as documented by Rapp and Cruz (1986).

[e]Height obtained using trigonometric method.

[f]The set 1 geoid height computation is under investigation. It seems inconsistent with nearby marks F 391 and Platteville.

Note: Standard Error of Unit Weight—Adjustment 1 (set 1 = 1.0; set 2 = 1.0); Adjustment 2 (set 1 = 1.2; set 2 = 1.5).

of GPS-derived orthometric heights. In order to fully understand the inconsistencies, a detailed investigation of geoid slope and changes in slope of the geoid should be performed and, where necessary, improved models used to estimate geoid height differences.

In addition to investigating geoid height differences, leveling height differences should be analyzed to ensure that a bench mark was not disturbed since the last time it was leveled. It is possible that significant discrepancies would be found if all bench marks were releveled. A true test of present GPS-derived ellipsoid height differences and geoid prediction accuracies is often possible only when GPS and leveling for a project occur very close together in time. The leveling data surrounding the Summit County GPS project were analyzed and all loop closures were within allowable tolerances. However, the bench marks could have been disturbed between the date the leveling data were obtained and the date the GPS survey was performed.

TABLE 7. Comparison of Heights in Boulder County, Colorado, GPS Survey Project: Rotation (Seconds of Arc) and Scale Parameters

Project location (1)	Set 1[a] (2)	Set 2[b] (3)	Sigma (4)
Western Half:			
North	4.48	−11.90	0.23
East	−0.05	−2.68	0.18
Azimuth	−1.63	−1.59	0.11
Scale (ppm)	−1.93	−1.36	1.19
Eastern Half:			
North	−0.21	0.25	0.24
East	−0.61	−1.36	0.14
Azimuth	−1.74	−1.40	0.22
Scale (ppm)	−0.57	−3.42	1.42

[a]Set 1—Geoid heights estimated using gravity data and Stokes' integration method as documented by Fury (1984).

[b]Set 2—Geoid heights based on Rapp's 360 × 360 model (OSU86F) as documented by Rapp and Cruz (1986).

TABLE 8. Comparison of Heights in Summit County, Ohio, GPS Survey Project

		$H_{Leveling}$ MINUS H_{GPS}					
		Adjustment 1[a]		Adjustment 2[b]		Geoid Height	
Station name (1)	$H_{Leveling}$ (m) (2)	Set 1[c] (cm) (3)	Set 2[d] (cm) (4)	Set 1[c] (cm) (5)	Set 2[d] (cm) (6)	Set 1 (m) (7)	Set 2 (m) (8)
Airport	318.491	−15.5	13.5	Constrained	Constrained	−31.8	−33.8
B 10	335.572	−27.5	11.5	Constrained	Constrained	−32.0	−33.9
Horst AZ MK	365.365	0.4	−0.6	2.9	12.3	−31.3	−33.6
H 179	295.862	2.2	8.2	9.1	8.6	−31.5	−33.7
Louisville[e]	387.8	−35.0	1.0	−3.4	−14.4	−31.9	−33.9
N 177	292.214	3.5	−1.5	5.7	3.4	−31.3	−33.7
Otis	348.030	−18.0	13.0	−6.5	0.3	−31.9	−33.8
Rumph[e]	392.2	−22.6	2.4	−7.5	−11.2	−31.7	−33.8
Square	355.033	−14.6	13.4	−2.0	0.2	−31.8	−33.8
S 337	332.237	−17.2	11.8	−3.1	−2.0	−31.8	−33.8
U 177	293.097	Constrained	Constrained	Constrained	Constrained	−31.4	−33.7
White Pond	299.651	−3.6	12.4	6.7	−4.2	−31.7	−33.9
39 RHP	362.138	−12.6	7.4	6.4	3.8	−31.6	−33.7

[a]Adjustment 1—Adjusted heights (H_{GPS}) obtained from 3-D minimum constraint least squares adjustment.

[b]Adjustment 2—Adjusted heights (H_{GPS}) obtained from 3-D least squares adjustment solving for geoidal tilt and scale.

[c]Set 1—Geoid heights estimated using gravity data and Stokes' integration method as documented by Fury (1984).

[d]Set 2—Geoid heights based on Rapp's 360 × 360 model (OSU86F) as documented by Rapp and Cruz (1986).

[e]Height obtained using trigonometric method.

Note: Standard Error of Unit Weight—Adjustment 1 (set 1 = 1.2; Set 2 = 1.2); Adjustment 2 (set 1 = 1.2; set 2 = 1.2).

TABLE 9. Comparison of Heights in Summit County, Ohio, GPS Survey Project: Rotation (Seconds of Arc) and Scale Parameters

Project Location (1)	Set 1[a] (2)	Set 2[b] (3)	Sigma (4)
North	1.53	−1.52	0.41
East	0.00	1.52	0.47
Azimuth	−1.06	−1.06	0.16
Scale (ppm)	0.58	0.19	0.39

[a]Set 1—Geoid heights estimated using gravity data and Stokes' integration method as documented by Fury (1984).

[b]Set 2—Geoid heights based on Rapp's 360 × 360 model (OSU86F) as documented by Rapp and Cruz (1986).

TABLE 10. Comparison of Heights in Boulder County, Colorado, GPS Survey Project Using Estimates of "True" Helmert Orthometric Heights

		$H_{Leveling}$ MINUS H_{GPS}					
		Adjustment 1[a]		Adjustment 2[b]		Geoid Height	
Station name (1)	$H_{Leveling}$ (m) (2)	Set 1[c] (cm) (3)	Set 2[d] (cm) (4)	Set 1[c] (cm) (5)	Set 2[d] (cm) (6)	Set 1 (m) (7)	Set 2 (m) (8)
Buff	1,637.778	12.7	−8.3	9.4	−1.1	−19.5	−15.5
Bunce	2,505.020	67.9	−120.1	−3.7	−4.4	−16.4	−14.1
Copeland	2,539.835	92.2	−147.8	Constrained	Constrained	−15.5	−13.7
Evergreen[e]	2,211.90	[e]	[e]	[e]	[e]	[e]	[e]
F 391	1,439.004	1.1	−64.9	23.8	−8.2	−21.7	−18.2
F 414	1,404.512	64.4[f]	−73.6	90.4[f]	−5.8	−21.4	−18.6
Ft. Collins 2	1,526.217	−60.5	−73.5	−30.9	−5.7	−20.1	−16.0
Granby[e]	2,491.30	[e]	[e]	[e]	[e]	[e]	[e]
Line	1,621.286	−20.8	6.2	−14.4	−0.7	−21.2	−16.7
J 320 Reset	1,532.455	−13.0	−6.0	−15.4	19.5	−20.4	−16.1
NOAA	1,697.150	5.1	−10.9	Constrained	Constrained	−19.5	−15.5
Peak	2,785.471	93.8	−113.2	Constrained	Constrained	−15.9	−13.8
Platteville	1,497.296	−6.3	−49.3	Constrained	Constrained	−21.8	−18.0
T 320 Reset	1,704.861	−15.5	−34.5	−21.8	4.3	−19.4	−15.4
TT 23 USGS	1,797.943	Constrained	Constrained	Constrained	Constrained	−19.8	−15.6
U 356	1,609.858	−50.2	−79.2	Constrained	Constrained	−19.6	−15.7
Ware GPS	1,603.694	14.1	−2.9	13.1	5.3	−19.5	−15.5
X 402	1,489.780	−60.4	−61.4	−23.1	7.4	−20.6	−16.4

[a]Adjustment 1—Adjusted heights (H_{GPS}) obtained from 3-D minimum constraint least squares adjustment.

[b]Adjustment 2—Adjusted heights (H_{GPS}) obtained from 3-D least squares adjustment solving for geoidal tilt and scale.

[c]Set 1-Geoid heights estimated using gravity data and Stokes' integration method as documented by Fury (1984).

[d]Set 2-Geoid heights based on Rapp's 360 × 360 model (OSU86F) as documented by Rapp and Cruz (1986).

[e]Height obtained using trigonometric method-not included.

[f]The set 1 geoid height computation is under investigation. It seems inconsistent with nearby marks F 391 and Platteville.

Note: Standard Error of Unit Weight—Adjustment 1 (set 1 = 1.0; set 2 = 1.0); Adjustment 2 (set 1 = 1.2; set 2 = 1.2).

TABLE 11. Comparison of Heights in Boulder County, Colorado, GPS Survey Project Using Estimates of "True" Helmert Orthometric Heights: Rotation (Seconds of Arc) and Scale Parameters

Project location (1)	Set 1[a] (2)	Set 2[b] (3)	Sigma (4)
Western Half:			
North	7.71	−8.71	0.23
East	−0.73	−3.36	0.18
Azimuth	−1.68	−1.65	0.11
Scale (ppm)	−2.00	−1.44	1.19
Eastern Half:			
North	0.22	0.68	0.24
East	−0.75	−1.51	0.14
Azimuth	−1.74	−1.40	0.22
Scale (ppm)	0.29	−2.55	1.42

[a]Set 1—Geoid heights estimated using gravity data and Stokes' integration method as documented by Fury (1984).

[b]Set 2—Geoid heights based on Rapp's 360 × 360 model (OSU86F) as documented by Rapp and Cruz (1986).

Analysis

The projects discussed have shown how GPS-derived orthometric height differences can vary considerably within the same survey. It is easy to understand that the main problem in estimating GPS-derived orthometric heights is estimating the accuracy of the value where there is no known orthometric height, which is exactly what the user needs. A better estimate of the shape of the geoid, as well as changes in the slope, must be obtained before GPS-derived orthometric heights can be routinely used by the surveying community. The authors believe that in certain areas of the country, GPS-derived orthometric heights can be estimated accurately enough to meet the needs of many users. The estimates, however, must be used with extreme caution because of the large uncertainty in the estimates of geoid height differences.

Tables 6 and 7 list the results from GPS projects performed in mountainous regions, i.e., Boulder County, Colorado. Fig. 3 depicts the Boulder County GPS network. Published orthometric heights based on normal gravity were used to estimate the leveling height differences for Tables 6, 7, 8 and 9. These heights, called normal orthometric heights, are published by NGS and available to all surveyors. Due to network adjustment procedures, published normal orthometric height differences could disagree with the observed leveling differences by several centimeters. For this report, these differences are considered to be insignificant because of the large uncertainties in the geoid height differences, although in Summit County, Tables 8 and 9, the results approach the 5–10 cm level of uncertainty.

From Tables 6 and 7, it appears that GPS-derived orthometric heights in mountainous regions may have uncertainties of +/− (0.5 m). In fact, uncertainties of 1.5 m were detected when geoid heights were estimated using Rapp's OSU86F model (n = 360 deg. harmonic series expansion) (Rapp and Cruz 1986). This is not meant to be a negative comment. The authors only

SUMMIT COUNTY, OHIO

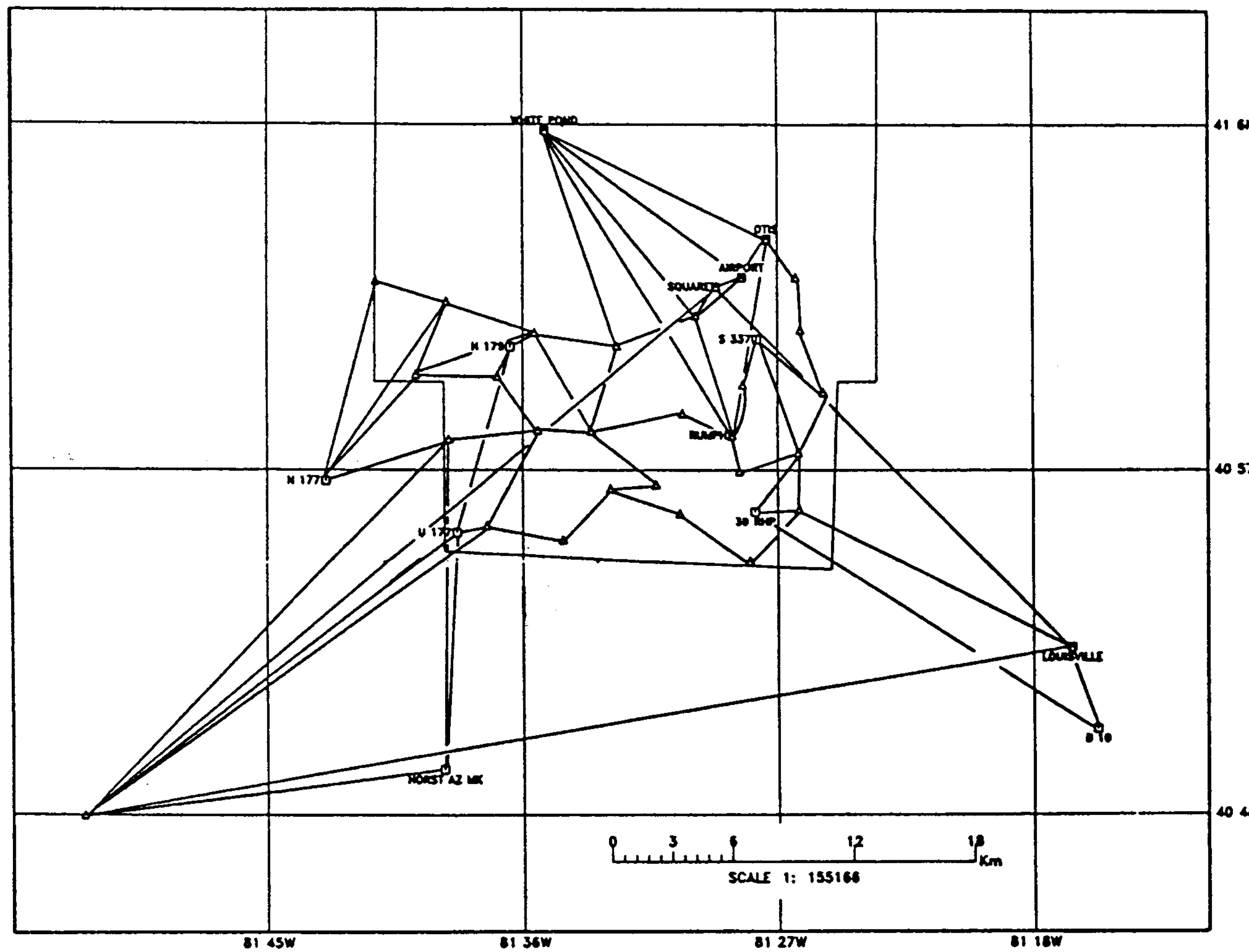

FIG. 2. Network Design of GPS Project in Summit County, Ohio

BOULDER COUNTY, CO.

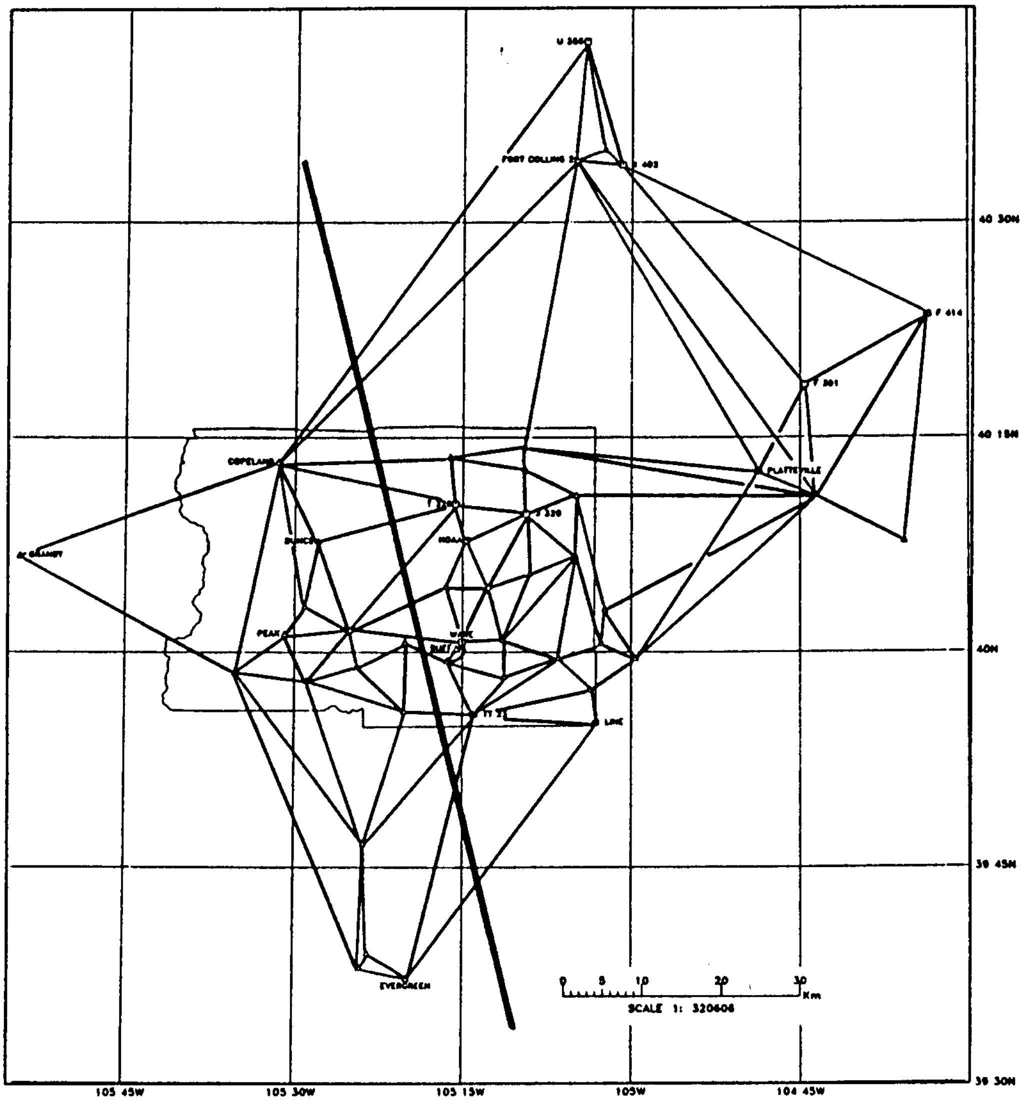

FIG. 3. Boulder County, Colorado GPS Network

wish to point out to users that every project must be designed properly and analyzed carefully. It is very important that bench marks (determined by leveling) are strategically located throughout the project.

Fig. 4 is a plot of geoid heights that were estimated using four techniques for five bench marks which were in the Boulder County GPS survey. The following four techniques were used: (1) Computation using the Earth's gravity field represented by spherical harmonic coefficients to order and degree 180 derived by Rapp (1981); (2) spherical harmonics to order and degree 360 derived by Rapp and Cruz (1986); (3) Stokes' integration procedure; and (4) geoid heights estimated using GPS and leveling data ($N = h - N$). Fig. 4 indicates that the spherical harmonic models only represent the long wavelength of the geoid and that Stokes' integration method using gravity data improves the estimates of geoid heights. Referring to Fig. 4, it is apparent why bench marks are required throughout the surveying project.

Even the best estimates of geoid heights usually have systematic errors

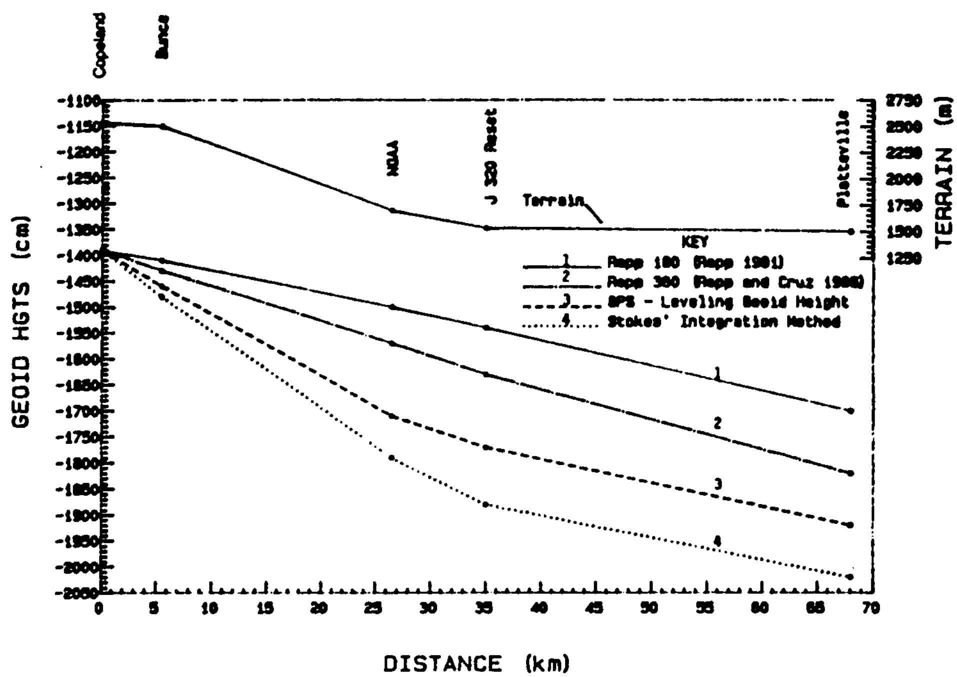

FIG. 4. Plot of Geoid Heights Estimated with Four Techniques for Five Marks in Boulder County GPS Survey

that are local in nature. These errors are in absolute magnitude as well as in tilt. Vincenty (1987a, 1987b) describes the mathematical models required to solve for these parameters. The geoidal slope is absorbed by two rotations (one around the north axis and the other around the east axis in the horizon system) and the geoidal heights are absorbed by the scale correction.

In order to evaluate the process, trend (bias) parameters were solved for in two networks: Boulder County and Summit County. Tables 6, 7, 8, and 9 contain the results of the adjustments. It should be noted that the Boulder County project was separated into two components. The boundaries of the components were based on the large change in geoid slope as indicated by comparing differential leveling orthometric heights to GPS-derived orthometric heights. It is important that enough leveling elevations are evenly distributed throughout the project in order to separate the network into trend (bias) groups that best represent the slope and changes in slope of the geoid (Vincenty 1987b). The points must be distributed in such a manner as to provide a strong determination of a plane. Three fixed elevations are required to solve for each additional set of parameters. Therefore, for every set of parameters, the network should have at least five marks with leveling elevations: three to solve for the parameters and two to check the results.

Looking at Tables 6, 7, 8 and 9, it is clear that solving for the parameters improved the overall estimates of GPS-derived orthometric heights. The majority of the differences between GPS-derived orthometric heights and published heights decreased. It should be noted that the Boulder County area is an extreme case. It does not represent typical GPS projects being performed in the United States. The results, however, are very encouraging and show that GPS-derived orthometric height determination deserves more attention in the future.

It was previously mentioned that due to network adjustment procedures, published normal orthometric height differences could disagree with the observed leveling data by several centimeters. It should also be noted that currently published NGS orthometric heights are based on normal gravity, not actual gravity values. The effect of using normal gravity instead of actual gravity to estimate orthometric differences will vary, depending on the project's location and its extent. Usually the effect is small, but it can be large in areas like Boulder County where large elevation differences occur over short distances.

The leveling network surrounding the Boulder County GPS network was generated using the latest data available. Geopotential differences were generated and validated, using gravity values derived from the SEG 4-km gridded Bouguer anomaly data set obtained from the Society of Exploration Geophysicists, loop misclosures were computed and checked against allowable tolerances, and a minimum constraint least squares adjustment was performed to estimate "true" Helmert orthometric heights. All data outliers were detected and removed.

Tables 10 and 11 contain the results using the heights estimated from the special leveling network adjustment. The results from the 3-D minimum constraint least squares GPS adjustments indicate that the differences in heights using geoid heights estimated using Stokes' integration method increased and the differences in heights using geoid heights estimated using OSU86F decreased. Once again, solving for the parameters improved the overall estimates of GPS-derived orthometric heights.

The minimum steps required when analyzing GPS-derived orthometric heights are as follows:

1. During the project's planning stage, perform a detailed analysis of the geoid in the area of the survey in order to determine if additional gravity and/or leveling data are required to adequately estimate the geoid slope and changes in slope.

   a. Perform a detailed study of the density and distribution of observed gravity values.

   b. Perform a detailed study of the leveling network in the area, i.e., plot all the leveling lines, note the age of leveling data, determine if bench marks can be occupied by GPS equipment, etc.

2. Perform a 3-D minimum constraint least-squares adjustment.
3. Compare the adjusted GPS-derived orthometric height differences obtained from step 2 with leveling-derived orthometric height differences.
4. Detect and remove all data outliers from steps 2 and 3 above.
5. Analyze the local geoid in detail.

   a. Plot geoid heights in the area.

   b. Plot estimated slope of geoid using differences between GPS-derived ellipsoid height differences and leveling-derived orthometric height differences ($dN = dh - dH$).

   c. Plot elevations versus geoid heights to see if the geoid heights differences appear to be consistent with surrounding values.

6. Estimate the local systematic errors in the geoid heights by solving for the geoidal slope and scale using the method described by Vincenty (1987*a*).
7. Compare the adjusted GPS-derived orthometric height differences from step 6 with the leveling-derived orthometric height differences.

TABLE 12. Comparison of Projected Subsidence Based on Historical Differential Leveling Measurements and Results of Measuring Vertical Height Motion (Subsidence) by GPS Height Difference Measurements in an Area Southeast of Phoenix, Arizona, Performed February 1984–May 1985 (15 Months Between Surveys) Using Single-Frequency Macrometer V1000 Survey Systems

Stations		Line length	Amount of Subsidence		
From (1)	To (2)	(km) (3)	GPS (*dh*) (cm) (4)	Projected (*dH*) (cm) (5)	Difference (cm) (6)
M279	A509	11	−11.2	−7.9	−3.3
H363	Q363	14	0.0	0.0	0.0
M279	D279	19	−8.5	−7.9	−0.6
H363	A279	21	0.0	−2.4	2.4
M279	A279	24	−3.0	−2.4	−0.6
M279	Q363	30	−0.5	0.0	−0.5
M279	H363	45	−1.7	0.0	−1.7

Estimating Vertical Crustal Motion Using GPS

This report has mentioned several times that the largest error source in estimating orthometric heights using GPS and gravity data is the inadequacy of the models to accurately represent the relief of the geoid. The main purpose of performing geodetic leveling is to estimate orthometric heights that are consistent with a particular datum, e.g., NGVD 29. Therefore knowing the uncertainties of the estimates of geoid height differences is critical.

Leveling is also used to estimate vertical crustal motion when two or more leveling surveys have been performed over some of the same bench marks, enabling one to estimate changes in height differences between bench marks over time. GPS satellite survey data can also be used to estimate vertical crustal movement.

Changes in ellipsoid heights determined from repeat GPS surveys can be evaluated independently of the geoid; i.e., the uncertainties associated with estimates for geoid height differences can be ignored. Thus, repeat GPS surveys can be used as an accurate alternative to repeat leveling surveys. Results obtained by Strange (1985) for a project southeast of Phoenix, Arizona, are listed in Table 12. Table 12 shows that it is possible to use multiple GPS occupations of the same point to estimate subsidence with uncertainties that are typically less than 2 cm over 20-kilometer distances.

Conclusions

Since early 1983, NGS has performed control survey projects in the United States using GPS satellities. These surveys have met the requirements of many users.

It is obvious that GPS-derived orthometric heights will have a major impact on the surveying community in the future. There are, however, several factors which need to be understood by users before GPS-derived orthometric heights can be routinely used by the surveying community. The user should perform a detailed analysis of the geoid in the area of the survey in order to determine if additional gravity data and/or leveling are required to adequately estimate the geoid's slope and its changes in slope. NGS is working on algorithms and models to improve the computation of geoid heights and geoid height differences. NGS is actively pursuing the Integrated Geodesy approach of combining leveling data and GPS measurements with gravity data to solve for an improved geoid (Hein 1985, Milbert and Holdahl 1988). Network design must include bench marks with leveling orthometric heights strategically located throughout the network in order to verify the estimates of geoid height differences. This may require obtaining additional leveling data in certain portions of the network where control is sparse or where orthometric heights are based on very old surveys.

Results of analyses performed by the authors show that orthometric heights in certain regions of the United States can be determined by GPS and gravity data with uncertainties between 5–15 cm. Analyses indicate that with appropriate planning, consideration of GPS survey specifications for connection to bench marks, proper field observing procedures, and a proper strategy for estimating geoid undulation differences and final orthometric height values, it is possible to use GPS survey methods to estimate orthometric heights to meet a wide range of engineering and land surveying requirements for vertical control.

Efforts will be made to improve the accuracies of geoid undulation differences dependent on overall national accuracy needs for determining GPS-derived orthometric heights and on costs of differential leveling versus GPS and gravity survey methods. Therefore, another question needs to be addressed. What are the accuracy requirements of most engineering and land surveying applications, as well as mapping applications? This is best answered by the users, and will influence how much effort should be directed toward estimating more accurate geoid values.

Appendix. References

Brennecke, J., et al. (1983). "A European astro-gravimetric geoid." *Reihe B: Angewandtw Geodäsie—Heft Nr. 269.*, Verlag des Instituts für Angewandte Geodäsie, Frankfort, FRG.

Engelis, T., Tscherning, C. C., and Rapp, R. H. (1984). "The precise computation of geoid undulation differences with comparison to results obtained from GPS." *Geophys. Res. Letters.*, 1(9), Sept., 821–824.

Fury, R. J. (1984). "Prediction of deflections of the vertical by gravimetric methods." *NOAA Technical Report NOS NGS 28*, National Geodetic Information Center, NOAA, Rockville, Md., 25 pp.

Fury, R. J. (1986). "Geoid height estimation technology development to improve computed orthometric height as a complement to GPS." Internal report, National Geodetic Survey, NOAA, Rockville, Md.

Hein, G. W. (1985). "Orthometric height determinations using GPS observations and integrated geodesy adjustment model." *NOAA Technical Report NOS 110 NGS 32*. National Geodetic Information Center, NOAA, Rockville, Md., 16 pp.

Heiskanen, W. A. and Moritz, H. (1981). *Physical geodesy*. Reprint, Institute of Physical Geodesy, Steyrergasse 17, A-8010, Graz, Austria.

Hothem, L. D. (1988). "Geometric survey standards and specifications for geodetic surveys using GPS relative positioning techniques." Version 5, May, National Geodetic Information Center, Rockville, Md., 50 pp.

Hothem, L. D., Goad, C. C., and Remondi, B. W. (1984): "GPS satellite surveying—practical aspects," *The Canadian Surveyor*. 38(3), Autumn, 177–192.

Kearsley, A. H. W. (1984). "Precision limitations and data requirements for the determination of relative geoid heights from gravimetry." *Report No. 26*, Dept. of Geodesy, Univ. of Uppsala, Institute of Geophysics, Sweden, 111 pp.

Kearsley, A. H. W. (1985). "Towards the optimum evaluation of the inner zone contribution to geoidal heights." *Aust. J. Geod. Photo. Surv.*, (42), June, 75–98.
Kearsley, A. H. W. (1986a). "The determination of precise geoid height differences using ring integration." *Proc. of International Symposium on the Definition of the Geoid.*, Florence, Italy, May 26–30, 151–174.
Kearsley, A. H. W. (1986b). "Data requirements for determining precise relative geoid heights from gravimetry." *J. of Geophysical Research.* 91(B9), 9193–9201.
Kearsley, A. H. W. (1987). "Tests on the recovery of precise geoid height differences from gravimetry, unpublished manuscript, School of Surveying of New South Wales, Kensington, NSW, Australia. 34 pp.
King, R. W., et al. (1985). *Surveying with GPS.* Monograph No. 9, School of Surveying, Univ. of South Wales, Kensington, NSW, Australia 2033, Nov., 138 pp.
Milbert, D. G., and Holdahl, S. R. (1988). "Combining leveling and GPS measurements through the integrated geodesy approach." *EOS. Transaction,* AGU, 69(16), 325, Apr. 19.
Moritz, H. (1983). "Local geoid determination in mountain regions." *Report No. 352,* Dept. of Geodetic Science and Surveying, Ohio State Univ., Columbus, Ohio, 47.
Proc. of the First International Symposium on Precise Positioning with the Global Positioning System (1985). National Geodetic Information Center, NOAA, Rockville, Md., Apr. 15–19, 946 pp.
Proc. of the Fourth International Geodetic Symp. on Satellite Positioning (1986). Defense Mapping Agency and National Geodetic Survey, Austin, Tex., Apr. 28–May 2.
Rapp, R. H. (1981). "The Earth's gravity field to degree and order 180 using Seasat altimeter data, terrestrial gravity data, and other data." *Report No. 322,* Dept. of Geodetic Science and Surveying, Ohio State Univ., Columbus, Ohio.
Rapp, R. H., and Cruz, J. Y. (1986). "Spherical harmonic expansions of the Earth's gravitational potential to degree 360 using 30′ mean anomalies." *Report No. 376,* Dept. of Geodetic Science and Surveying, Ohio State Univ., Columbus, Ohio.
Remondi, R. W. (1984). "Using the Global Positioning System (GPS) phase observable for relative geodesy: modeling, processing, and results," thesis presented to the University of Texas, at Austin, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
Standards and specifications for geodetic control networks (1984). Federal Geodetic Control Committee, National Geodetic Information Center, NOAA, Rockville, Md., 33 pp.
Strange, W. E. (1985). "High-precision, three-dimensional differential positioning using GPS." *Proc. of the First Int. Symp. on Precise Positioning with the Global Positioning System,* Rockville, Md., Apr. 15–19, 543–548.
Schwarz, K. P., Sideris, M. G., and Forsberg, R. (1987). "Orthometric heights without leveling." *J. Surv. Engrg.*, ASCE, 113(1), 28–40.
Tscherning, C. C. (1974). "A FORTRAN IV program for the determination of the anomalous potential using stepwise least squares collocation." *Report No. 212,* Dept. of Geodetic Science and Surveying, Ohio State Univ., Columbus, Ohio.
Vanicek, P. and John, S. (1983). "Evaluation of geoid solutions for Canada using different kinds of data," presented to IAG Symposium on Improved Gravity Field Estimations on Global Basis, Hamburg, F.R.G., Aug. 19.
Vincenty, T. (1987a). "On the use of GPS vectors in densification adjustment." *Surveying and Mapping,* 47(2), 103–108.
Vincenty, T. (1987*b*). "Geoid heights for GPS densifications." *ACSM Bulletin,* Dec.
Wells, D. (1986). *Guide to GPS Positioning,* Prepared by Canadian GPS Associates, ISBN: 0-920-114-73-3, P.O. Box 3184, Postal Station B, Fredericton, New Brunswick E3A 5G9, Canada, 650 pp.